Canonical Mapping of O(N) Vector /Higher Spin Correspondence

BROWN

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Higher Spin Gravity

AdS$_4$ Higher Spin Theory

- M. A. Vasiliev
- Infinite tower of higher spin

CFT$_3$: O(N) , U(N) vector

- Boson : φ_i (i=1,…,N)
- Singlet sector

Objective

- Minimal Bosonic Theory S=0,2,4,… ⟷ O(N) real vector model

Spin Theo

- In the bulk with emergent AdS dimension

- Non-minimal Bosonic Theory S=0,1,2,… ⟷ U(N) complex vector model

(covered

Contents

- 1. Review : Map in Light-cone Gauge

  [A. Jevicki, J.P.Rodrigues, R.d.M.Koch, K.Jin, 2010]

- 2. Extention to Time-like Gauge

  [JY, A. Jevicki, J.P.Rodrigues, R.d.M.Koch, 2014]

  - Quick Derivation of Bi-local Map
  - SO(2,3) generator for HS field and Bi-local Map

HS Field : Light-cone Gauge

- Spin-s field : package with harmonic oscillator

$$\Phi = \Phi^{A_1 \cdots A_s} \alpha_{A_1} \cdots \alpha_{A_s} \qquad \Phi^{A_1 \cdots A_s} = e^{A_1}_{\mu_1} \cdots e^{A_s}_{\mu_s} h^{\mu_1 \cdots \mu_s} \qquad [\bar{\alpha}_A, \alpha_B] = \eta_{AB}$$

- Symmetric, double-traceless

$$(\bar{\alpha}^A \bar{\alpha}_A)^2 \Phi = 0 \qquad \alpha^A \bar{\alpha}_A \Phi = s\Phi$$

- Light-cone gauge : $\bar{\alpha}^+ \Phi = 0$
- Physical Field : two helicities

$$\Phi_{\text{phy}} = \Phi^{I_1 \cdots I_s}_{\text{phy}} \alpha_{I_1} \cdots \alpha_{I_s} \qquad (I = 1, 2)$$ [R.R.Metsaev, 1999]

$$\bar{\alpha}^I \bar{\alpha}_I \Phi_{\text{phy}} = 0 \qquad \alpha^A \bar{\alpha}_A \Phi_{\text{phy}} = s\Phi_{\text{phy}} \qquad \partial^\mu \partial_\mu \Phi_{\text{phy}} = 0$$

Spin matrix $M_{12} = \alpha_1 \bar{\alpha}_2 - \alpha_2 \bar{\alpha}_1 \longrightarrow S^1$

$$\Phi_{\text{phy}} = \Phi_s \beta_+^s + \bar{\Phi}_s \beta_-^s \qquad \beta_\pm = \alpha_1 \pm i\alpha_2$$

HS SO(2,3) Generators : Light-cone

- SO(2,3) generator for physical HS field on $AdS_4 \times S^1$ (in Light-cone gauge) [R.R.Metsaev, 1999]

Poincare

$$\hat{p}^- = -\frac{p^x p^x + p^z p^z}{2p^+} \qquad \hat{m}^{+-} = -x^- p^+$$

$$\hat{p}^+ = p^+ \qquad \hat{m}^{+x} = -x p^+$$

$$\hat{p}^x = p^x \qquad \hat{m}^{-x} = x^- p^x - x\hat{p}^- + \frac{p^\theta p^z}{p^+}$$

Dilation

$$\hat{d} = x^- p^+ + x p^x + z p^z + 1$$

Special Conformal

$$\hat{k}^- = -\frac{1}{2}(x^2 + z^2)\hat{p}^- + x^- \hat{d} + \frac{1}{p^+}((x p^z - z p^x) p^\theta + (p^\theta)^2)$$

$$\hat{k}^x = -\frac{1}{2}(x^2 + z^2) p^x + x\hat{d} + z p^\theta$$

$$\hat{k}^+ = -\frac{1}{2}(x^2 + z^2) p^+$$

Schematic Bi-local Map

- SO(2,3) Decomposition [C. Fronsdal & M. Flato, 1978]

$$D(1/2,0)\otimes D(1/2,0)=\sum_{s=0}^{\infty} D\left(s+1,s\right)$$ cf. Addition of Angular momentum

- Bi-local Field : O(N) invariants

$$\Psi(x^+;x_1^-,x_1,x_2^-,x_2)=\vec{\varphi}(x^+;x_1^-,x_1)\cdot\vec{\varphi}(x^+;x_2^-,x_2)$$

  - SO(2,3) Generators for Bi-local field

- Map form collective field to HS field

$$H\left(X,y\right)=\int du_1 du_2\, K\left(X,y|u_1,u_2\right)\Psi\left(u_1,u_2\right)$$

Bi-local Generator : Light-cone Gauge

- SO(2,3) Bi-local generator

Poincare

$$\hat{p}^- = -\frac{p_1 p_1}{2p_1^+} - \frac{p_2 p_2}{2p_2^+} \qquad \hat{m}^{+x} = -x_1 p_1^+ - x_2 p_2^+$$

$$\hat{p}^+ = p_1^+ + p_2^+ \qquad \hat{m}^{-x} = x_1^- p_1 + x_2^- p_2 - x_1\left(-\frac{p_1 p_1}{2p_1^+}\right) - x_2\left(-\frac{p_2 p_2}{2p_2^+}\right)$$

$$\hat{p}^x = p_1 + p_2 \qquad \hat{m}^{+-} = -x_1^- p_1^+ - x_2^- p_2^+$$

Dilation

$$\hat{d} = x_1^- p_1^+ + x_1 p_1 + x_2^- p_2^+ + x_2 p_2 + 1$$

Special Conformal

$$\hat{k}^+ = -\frac{1}{2}x_1 x_1 p_1^+ - \frac{1}{2}x_2 x_2 p_2^+$$

$$\hat{k} = -\frac{1}{2}x_1 x_1 p_1 + x_1(x_1^- p_1^+ + x_1 p_1 + \frac{1}{2}) - \frac{1}{2}x_2 x_2 p_2 + x_2(x_2^- p_2^+ + x_2 p_2 + \frac{1}{2})$$

$$\hat{k}^- = -\frac{1}{2}x_1 x_1\left(-\frac{p_1 p_1}{2p_1^+}\right) + x_1^-(x_1^- p_1^+ + x_1 p_1 + \frac{1}{2}) - \frac{1}{2}x_1 x_1\left(-\frac{p_1 p_1}{2p_1^+}\right) + x_2^-(x_2^- p_2^+ + x_2 p_2 + \frac{1}{2})$$

Map : Canonical Transformation

- Identifying two generator $J_{\text{HS}} = J_{\text{Bi-local}} = J^1_{CFT} + J^2_{CFT}$
- Bi-local Map [A. Jevicki, J.P.Rodrigues, R.d.M.Koch, K.Jin, 2010]

AdS — CFT

$$p^+ = p_1^+ + p_2^+$$

$$p = p_1 + p_2$$

$$p^z = p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}}$$

$$\theta = 2\arctan\sqrt{\frac{p_2^+}{p_1^+}}$$

AdS — CFT

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}$$

$$x = \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+}$$

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}$$

$$p^\theta = \sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{x_1 - x_2}{2}\left(p_1\sqrt{\frac{p_2^+}{p_1^+}} + p_2\sqrt{\frac{p_1^+}{p_2^+}}\right)$$

- Canonical Transformation

$$\{p^+, x^-\}_{\text{Bi-local PB}} = \{p^x, x\}_{\text{Bi-local PB}} = \{p^z, z\}_{\text{Bi-local PB}} = \{p^\theta, \theta\}_{\text{Bi-local PB}} = 1$$

Kernel

- Kernel : $\mathcal{K}(p^+,p,p^z,\theta;p_1^+,p_1,p_2^+,p_2) = \mathcal{J}\delta(p_1^+ + p_2^+ - p^+)\delta(p_1+p_2-p)$
$$\times\delta(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z)\delta(2\arctan\sqrt{\frac{p_2^+}{p_1^+}} - \theta)$$
- One-to-one, induce Extra Bulk coordinate z
- Transformation : From Bi-local field to HS field

$$\widetilde{\mathcal{H}}(p^+,p,p^z,\theta) = \int dp_1^+ dp_1 dp_2^+ dp_2\ \mathcal{K}(p^+,p,p^z,\theta;p_1^+,p_1,p_2^+,p_2)\widetilde{\Psi}(p_1^+,p_1,p_2^+,p_2)$$

- Local

$$\widetilde{\mathcal{H}}(p^+,p,p^z,\theta) = \int dp_1^+ dp_1 dp_2^+ dp_2\ \mathcal{K}\ \widetilde{\Psi}$$
$$\widetilde{\mathcal{W}}(p^+,p,p^z,\theta) = \int dp_1^+ dp_1 dp_2^+ dp_2\ \mathcal{K}\ \widetilde{\Pi}$$

$$\left[\widetilde{\Psi}(p_1^+,p_1,p_2^+,p_2),\widetilde{\Pi}(k_1^+,k_1,k_2^+,k_2)\right] = p_1^+p_2^+\delta(p_1^+-k_1^+)\delta(p_1-k_1)\delta(p_2^+-k_2^+)\delta(p_2-k_2)$$

⇩

$$\left[\widetilde{\mathcal{H}}(p^+,p,p^z,\theta),\widetilde{\mathcal{W}}(k^+,k,k^z,\phi)\right] = p^+\delta(p^+-k^+)\delta(p-k)\delta(p^z-k^z)\delta(\theta-\phi)$$

Spin-s Primary Current

- Fourier Transformation : $(p^+, p, p^z, \theta) \longrightarrow (x^-, x, z, p^\theta = s)$

$$\mathcal{H}_s(x^+, x^-, x, z) = \int_{-2p^+p^- - p^2 > 0} dp^- dp^+ dp\, e^{ix^+p^- + ix^-p^+ + ixp} J_{-\frac{1}{2}}\left(z\sqrt{-2p^+p^- - p^2}\right)$$

$$\times \sqrt{\frac{\pi z \sqrt{-2p^+p^- - p^2}}{2}} \frac{s!}{\Gamma\left(s + \frac{1}{2}\right)(p^+)^s} \widetilde{\mathcal{O}}_s(p^-, p^+, p)$$

[I.Bena, 1999]

$z \longrightarrow 0$

$$\frac{s!}{\Gamma\left(s + \frac{1}{2}\right)\partial_-^s} \int_{-2p^+p^- - p^2 > 0} dp^- dp^+ dp\, e^{ix^+p^- + ix^-p^+ + ixp} \widetilde{\mathcal{O}}_s(x^-, x^+, x)$$

$$\mathcal{O}_s = \sum_{n=0}^{s/2} \frac{(-4)^n}{(2n)!} \sum_{k=0}^{s-2n} \frac{(-1)^k}{k!(s-2n-k)!} \partial_-^{n+k}\vec{\phi} \cdot \partial_-^{s-n-k}\vec{\phi}$$

: Spin-s primary field in CFT

- Bi-local map (in z → 0 limit) gives primary CFT operator

Time-like Gauge : Quick Derivation

- Transformation in "Momentum" space

$$(\vec{p}_1, \vec{p}_2) \longrightarrow (\vec{p}, p^z, \theta)$$

- Total momentum (CM of two particles)

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \qquad\qquad p^0 = |\vec{p}_1| + |\vec{p}_2|$$

- On-shell condition

$$p^z = \pm\sqrt{(p^0)^2 - \vec{p}^2} = 2\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right)$$

- SO(2,3) Casimir

$$\mathcal{C}_{SO(2,3)} = E_0^2 + s^2 \xrightarrow[\;]{\text{Massless representation}\; E_0 = s+1} \mathcal{C}_{SO(2,3)} = 2s^2 = 2(p^\theta)^2$$

- Compare to SO(2,3) Casimir of Bi-local generator

$$p^\theta = \sqrt{|\vec{p}_1|\,|\vec{p}_2|}\cos\frac{\varphi_1 + \varphi_2}{2}\,(x_2^1 - x_1^1) + \sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\frac{\varphi_1 + \varphi_2}{2}\,(x_2^2 - x_1^2)$$

Time-like Gauge : Quick Derivation

- Ansatz for θ (to have canonical transformation,)

$$\theta = \arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{(|\vec{p}_2| - |\vec{p}_1|)p^z}\right)$$

  - Can be found naturally.(Later)

- Bi-local Map in time-like gauge

$$\mathcal{K}(\vec{p}, p^z, \theta; \vec{p}_1, \vec{p}_2) = \mathcal{J}(\vec{p}_1, \vec{p}_2)\delta^{(2)}(\vec{p}_1 + \vec{p}_2 - \vec{p})\delta\left(2\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right) - p^z\right)$$
$$\times\delta\left(\arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{(|\vec{p}_2| - |\vec{p}_1|)p^z}\right) - \theta\right)$$

Coordinates Transformation

- "Momentum" space transformation

  → Coordinates transformation by Chain rule

$$x^1 = \frac{|\vec{p}_1|\, x_1^1 + |\vec{p}_2|\, x_2^1}{\sqrt{(p^z)^2 + \vec{p}^2}} - \frac{p^2 p^z p^\theta}{\vec{p}^2 \sqrt{(p^z)^2 + \vec{p}^2}}$$

$$x^2 = \frac{|\vec{p}_1|\, x_1^2 + |\vec{p}_2|\, x_2^2}{\sqrt{(p^z)^2 + \vec{p}^2}} + \frac{p^1 p^z p^\theta}{\vec{p}^2 \sqrt{(p^z)^2 + \vec{p}^2}}$$

$$z = \frac{(\vec{x}_1 - \vec{x}_2) \cdot \vec{p}_1\, |\vec{p}_2| - (\vec{x}_1 - \vec{x}_2) \cdot \vec{p}_2\, |\vec{p}_1|}{p^z\, (|\vec{p}_1| + |\vec{p}_2|)}$$

$$p^\theta = \sqrt{|\vec{p}_1|\, |\vec{p}_2|} \cos\frac{\varphi_1 + \varphi_2}{2}\, (x_2^1 - x_1^1) \; + \sqrt{|\vec{p}_1|\, |\vec{p}_2|} \sin\frac{\varphi_1 + \varphi_2}{2}\, (x_2^2 - x_1^2)$$

- Canonical Transformation by Construction

$$\{p^i, x^j\} = \delta^{ij} \qquad \{p^z, z\} = \{p^\theta, \theta\} = 1$$

Reduction

- Package spin s current with α

$$\mathcal{O}_s(x;\alpha) = \mathcal{O}^s_{\mu_1\mu_2\cdots\mu_s}\alpha^{\mu_1}\cdots\alpha^{\mu_s} \qquad [\bar{\alpha}^\mu, \alpha^\nu] = \eta^{\mu\nu}$$

- Constraints

$$\bar{\alpha}^\mu \partial_\mu \mathcal{O}_s(x;\alpha) = 0 \qquad \text{: Conservation}$$

$$\bar{\alpha}^\mu \bar{\alpha}_\mu \mathcal{O}_s(x;\alpha) = 0 \qquad \text{: Traceless}$$

$$\alpha^\mu \bar{\alpha}_\mu \mathcal{O}_s(x;\alpha) = s\mathcal{O}^s(x;\alpha) \qquad \text{: Spin s}$$

- To make contact with physical collective picture, we need to solve constraints
- Let's solve

Solve Constraints : Canonical Transformation

- (6+6) dim phase space $(x^\mu, p^\mu; \alpha^\mu, \bar{\alpha}^\mu)$
- Canonical Transformation

$$\bar{\alpha}^\mu \partial_\mu \mathcal{O}_s(x;\alpha) = 0$$
$$\bar{\alpha}^\mu \bar{\alpha}_\mu \mathcal{O}_s(x;\alpha) = 0$$

$$p^\mu \bar{\alpha}_\mu = 0$$
$$\bar{\alpha}^\mu \bar{\alpha}_\mu = 0$$

$(x^\mu, p^\mu; \alpha^\mu, \bar{\alpha}^\mu)$

T_1

$(\bar{x}^\mu, \bar{p}^\mu; \beta^\mu, \bar{\beta}^\mu)$

$$(\bar{\beta}^1)^2 + (\bar{\beta}^2)^2 - \frac{1}{(\bar{p}^0)^2}(\bar{\beta}^0\bar{p}^0 + \bar{\beta}^1\bar{p}^1 + \bar{\beta}^2\bar{p}^2)^2 = 0$$
$$-\bar{\beta}^0\bar{p}^0 = 0$$

T_2

$(\widetilde{x}^\mu, \widetilde{p}^\mu; \gamma^\mu, \bar{\gamma}^\mu)$

$$2\bar{\gamma}^1\bar{\gamma}^2 - \bar{\gamma}^0 f(\widetilde{p}, \bar{\gamma}) = 0$$
$$-\bar{\gamma}^0\widetilde{p}^0 = 0$$

Two Solutions

$$\gamma^0 = \bar{\gamma}^0 = \gamma^2 = \bar{\gamma}^2 = 0$$

$$\gamma^0 = \bar{\gamma}^0 = \gamma^1 = \bar{\gamma}^1 = 0$$

(Unconstrained) Primary spin-s Current

- (Unconstrained) Primary spin-s operator ($p^0 > 0$)

$$\widetilde{\mathcal{O}}_s(p;\epsilon) = \int d\vec{p}_1 d\vec{p}_2\ \delta^{(3)}\left(p_1^\mu + p_2^\mu - p^\mu\right) \frac{(-1)^s 2^{\frac{3}{2}s-1}}{s!}$$

$$\times \left(2\left|\vec{p}_1\right|\left|\vec{p}_2\right| - 2\vec{p}_1\cdot\vec{p}_2\right)^{\frac{s}{2}} e^{is\Theta(\vec{p}_1,\vec{p}_2)} a(p_1,p_2)$$

: annihilation operator of bi-local field

$$\widetilde{\mathcal{O}}_s(p;\epsilon^*) = \int d\vec{p}_1 d\vec{p}_2\ \delta^{(3)}\left(p_1^\mu + p_2^\mu - p^\mu\right) \frac{(-1)^s 2^{\frac{3}{2}s-1}}{s!}$$

$$\times \left(2\left|\vec{p}_1\right|\left|\vec{p}_2\right| - 2\vec{p}_1\cdot\vec{p}_2\right)^{\frac{s}{2}} e^{-is\Theta(\vec{p}_1,\vec{p}_2)} a(p_1,p_2)$$

$$\Theta\left(\vec{p}_1,\vec{p}_2\right) \equiv 2\arctan\left(\frac{2\vec{p}_1\times\vec{p}_2}{\sqrt{2\sqrt{\vec{p}_1^2\vec{p}_2^2} - 2\vec{p}_1\cdot\vec{p}_2}\left(\sqrt{\vec{p}_1^2} - \sqrt{\vec{p}_2^2}\right)}\right)$$

: Equal to θ map (up to sign)

$$\epsilon(p) \equiv \frac{1}{\sqrt{2}\,|\vec{p}|}\left(\frac{\vec{p}^2}{\sqrt{-p^\mu p_\mu}}, \frac{p^0p^1}{\sqrt{-p^\mu p_\mu}} + ip^2, \frac{p^0p^2}{\sqrt{-p^\mu p_\mu}} - ip^1\right)$$

: polarization vector

SO(2,3) Generator for HS field

- SO(2,3) Generator for physical spin-s primary

$$\hat{p}^{\mu} = p^{\mu}$$
$$\hat{d} = -x^0 p^0 + x^1 p^1 + x^2 p^2 + 2\alpha^1 \bar{\alpha}^1$$
$$\hat{m}^{12} = x^1 p^2 - x^2 p^1$$
$$\hat{m}^{20} = x^2 p^0 - x^0 p^2 + \left(\frac{p^2}{p^0} + i\frac{p^1}{q} - i\frac{(p^0)^2 p^1}{q\vec{p}^2}\right)\alpha^1\bar{\alpha}^1$$
$$\hat{k}^0 = -\frac{1}{2}(x^\mu x_\mu)p^0 + x^0\hat{D} + \frac{1}{2}(\alpha^1\bar{\alpha}^1)^2\left(\frac{q^2}{p^0\vec{p}^2} - \frac{3}{p^0} - \frac{\vec{p}^2}{p^0 q^2}\right)$$
$$-(\alpha^1\bar{\alpha}^1)\left(\frac{x^1p^1 + x^2p^2}{p^0} + i\frac{q}{\vec{p}^2}(x^1p^2 - x^2p^1)\right)$$

$$(p^0, p^1, p^2) \Rightarrow (p^1, p^2, p^z = \sqrt{(p^0)^2 - \vec{p}^2})$$

$$\hat{p}^0 = \sqrt{\vec{p}^2 + (p^z)^2}$$
$$\hat{p}^1 = p^1$$
$$\hat{p}^2 = p^2$$
$$\hat{m}^{01} = -x^1\hat{p}^0 - \frac{p^2 p^z}{\vec{p}^2}p^\theta$$
$$\hat{m}^{12} = x^1 p^2 - x^2 p^1$$
$$\hat{m}^{20} = x^2\hat{p}^0 - \frac{p^1 p^z}{\vec{p}^2}p^\theta$$
$$\hat{d} = x^1 p^1 + x^2 p^2 + z p^z$$
$$\hat{k}^0 = -\frac{1}{2}(\vec{x}^2 + z^2)\hat{p}^0 - \frac{p^z}{\vec{p}^2}\hat{m}^{12}p^\theta - \frac{\hat{p}^0}{2\vec{p}^2}(p^\theta)^2$$
$$\hat{k}^1 = -\frac{1}{2}(\vec{x}^2 + z^2)p^1 + x^1 D + \frac{z p^2 \hat{p}^0}{\vec{p}^2}p^\theta + \frac{p^1}{2\vec{p}^2}(p^\theta)^2$$
$$\hat{k}^2 = -\frac{1}{2}(\vec{x}^2 + z^2)p^2 + x^2 D - \frac{z p^1 \hat{p}^0}{\vec{p}^2}p^\theta + \frac{p^2}{2\vec{p}^2}(p^\theta)^2$$

HS Generator

Bi-local Map in Time-like Gauge

- HS generator = Bi-local generator
- Same Result

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \qquad p^z = 2\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right)$$

$$p^\theta = \sqrt{|\vec{p}_1|\,|\vec{p}_2|}\cos\frac{\varphi_1 + \varphi_2}{2}\,(x_2^1 - x_1^1)$$

$$x^1 = \frac{|\vec{p}_1|\,x_1^1 + |\vec{p}_2|\,x_2^1}{\sqrt{(p^z)^2 + \vec{p}^2}} - \frac{p^2 p^z p^\theta}{\vec{p}^2\sqrt{(p^z)^2 + \vec{p}^2}}$$

$$x^2 = \frac{|\vec{p}_1|\,x_1^2 + |\vec{p}_2|\,x_2^2}{\sqrt{(p^z)^2 + \vec{p}^2}} + \frac{p^1 p^z p^\theta}{\vec{p}^2\sqrt{(p^z)^2 + \vec{p}^2}}$$

$$z = \frac{(\vec{x}_1 - \vec{x}_2)\cdot\vec{p}_1\,|\vec{p}_2| - (\vec{x}_1 - \vec{x}_2)\cdot\vec{p}_2\,|\vec{p}_1|}{p^z\,(|\vec{p}_1| + |\vec{p}_2|)}$$

$$p^\theta = \sqrt{|\vec{p}_1|\,|\vec{p}_2|}\cos\frac{\varphi_1 + \varphi_2}{2}\,(x_2^1 - x_1^1) + \sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\frac{\varphi_1 + \varphi_2}{2}\,(x_2^2 - x_1^2)$$
$$+\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\frac{\varphi_1 + \varphi_2}{2}\,(x_2^2 - x_1^2)$$

Properties of Kernel

- "Momentum" space transformation : Kernel

$$\mathcal{K}(\vec{p}, p^z, \theta; \vec{p}_1, \vec{p}_2) = \mathcal{J}(\vec{p}_1, \vec{p}_2)\delta^{(2)}(\vec{p}_1 + \vec{p}_2 - \vec{p})\delta\left(2\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right) - p^z\right)$$
$$\times\delta\left(\arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{(|\vec{p}_2| - |\vec{p}_1|)p^z}\right) - \theta\right)$$
$$= \delta^{(2)}(\vec{p}_1 - \vec{\beta}_1(\vec{p}, p^z, \theta))\delta^{(2)}(\vec{p}_2 - \vec{\beta}_2(\vec{p}, p^z, \theta))$$

  - One-to-one

- Inverse kernel

$$\mathcal{Q}(\vec{p}_1, \vec{p}_2; \vec{p}, p^z, \theta) = \delta^{(2)}(\vec{p}_1 + \vec{p}_2 - \vec{p})\delta\left(2\sqrt{|\vec{p}_1|\,|\vec{p}_2|}\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right) - p^z\right)$$
$$\times\delta\left(\arctan\left(\frac{2\vec{p}_2 \times \vec{p}_1}{(|\vec{p}_2| - |\vec{p}_1|)p^z}\right) - \theta\right)$$

Properties of Kernel

- Local

$$\widetilde{\mathcal{H}}(\vec{p}, p^z, \theta) = \int d\vec{p}_1 d\vec{p}_2 \; \mathcal{K}(\vec{p}, p^z, \theta; \vec{p}_1, \vec{p}_2) \widetilde{\Psi}(\vec{p}_1, \vec{p}_2)$$

$$\widetilde{\mathcal{W}}(\vec{p}, p^z, \theta) = \int d\vec{p}_1 d\vec{p}_2 \; \mathcal{K}(\vec{p}, p^z, \theta; \vec{p}_1, \vec{p}_2) \widetilde{\Pi}(\vec{p}_1, \vec{p}_2)$$

$$\left[\widetilde{\Psi}(\vec{p}_1, \vec{p}_2), \widetilde{\Pi}(\vec{k}_1, \vec{k}_2)\right] = |\vec{p}_1| \, |\vec{p}_2| \, \delta^{(2)}(\vec{p}_1 - \vec{k}_1) \delta^{(2)}(\vec{p}_2 - \vec{k}_2)$$

⇓

$$\left[\widetilde{\mathcal{H}}(\vec{p}, p^z, \theta), \widetilde{\mathcal{W}}(\vec{k}, k^z, \phi)\right] = \sqrt{\vec{p}^2 + (p^z)^2} \; \delta^{(2)}(\vec{p} - \vec{k}) \delta(p^z - k^z) \delta(\theta - \phi)$$

- Redundancy in Bi-local field $\widetilde{\Psi}(\vec{p}_1, \vec{p}_2) = \widetilde{\Psi}(\vec{p}_2, \vec{p}_1)$

⇒ Redundancy in HS field $\widetilde{\mathcal{H}}(\vec{p}, p^z, \theta) = \widetilde{\mathcal{H}}(\vec{p}, -p^z, -\theta)$

$\cos s\theta \cos p^z z$ $\quad$ $\sin s\theta \sin p^z z$ $\quad$ ~~$\sin s\theta \cos p^z z$~~ $\quad$ ~~$\cos s\theta \sin p^z z$~~

$$\mathcal{H}_s^{(+)}(t, \vec{x}, z) \sim \cos s\theta \cos p^z z$$

$$\mathcal{H}_s^{(-)}(t, \vec{x}, z) \sim \sin s\theta \sin p^z z \qquad \mathcal{H}_0^{(-)} = 0$$

From CFT Primary to HS Field

- Kernel : creation(annhilation) operpator of bi-local field to HS creation(annihilation) operator
- Quadratic Hamiltonian of Collective field is mapped to Quadratic Hamiltonian of HS field

$$H_2 = \frac{1}{2}\int_{\vec{p}_1 \geq \vec{p}_2} d\vec{p}_1 d\vec{p}_2 \left[\widetilde{\Pi}'(\vec{p}_1, -\vec{p}_2)\widetilde{\Pi}'(\vec{p}_2, -\vec{p}_1) + \omega^2_{\vec{p}_1,\vec{p}_2}\widetilde{\Psi}'(\vec{p}_1, -\vec{p}_2)\widetilde{\Psi}'(\vec{p}_2, -\vec{p}_1)\right]$$

$$\Downarrow$$

$$H_2 = \int d\vec{p}dp^z d\theta \sqrt{\vec{p}^2 + (p^z)^2}\mathcal{A}^\dagger(\vec{p}, p^z, \theta)\mathcal{A}(\vec{p}, p^z, \theta)$$

From CFT Primary to HS Field

- Fourier Transformation : $(\vec{p}, p^z, \theta) \Longrightarrow (\vec{x}, z, p^\theta = s)$
- New basis

$$\mathcal{H}_s^{(\pm)}(\vec{x}, z) \equiv \frac{1}{2}\left[\mathcal{H}_s(\vec{x}, z) \pm \mathcal{H}_{-s}(\vec{x}, z)\right]$$

- Perform time-evolution and change of variable
  - Recall time-evolution of HS field is induced from time-evolution of bi-local field
  - From p^z to $p^0 = \sqrt{\vec{p}^2 + (p^z)^2}$

$$\mathcal{H}_s^{(\pm)}(t; \vec{x}, z) = \int d\vec{p} dp^z \cdots = \int d\vec{p} dp^0 \cdots$$

Summary : Time-like Map

- Canonical Map to Bulk field

$$\mathcal{H}_s^{(+)}(t;\vec{x},z) = \frac{\sqrt{\pi}s!}{2^{\frac{3}{2}s+\frac{3}{2}}}\int_{p^0>|\vec{p}|}\frac{d^2\vec{p}dp^0}{(2\pi)^3 2p^0}e^{-ip^0t+i\vec{p}\cdot\vec{x}}z^{\frac{1}{2}}\left[(p^0)^2-\vec{p}^2\right]^{-\frac{s}{2}+\frac{1}{4}}$$
$$\times J_{-\frac{1}{2}}(\sqrt{(p^0)^2-\vec{p}^2}\;z)\left[\widetilde{\mathcal{O}}_s(p;\epsilon)+\widetilde{\mathcal{O}}_s(p;\epsilon^*)\right]+\text{h.c}$$

$$\xrightarrow{z\longrightarrow 0}\quad 2^{-\frac{3}{2}s-1}s!\left(-(\partial^0)^2+\vec{\partial}^2\right)^{-\frac{s}{2}}\int_{p^0>|\vec{p}|}\frac{d^2\vec{p}dp^0}{(2\pi)^3 2p^0}\left[\widetilde{\mathcal{O}}_s(p;\epsilon)+\widetilde{\mathcal{O}}_s(p;\epsilon^*)\right]+\text{h.c.}$$

$$\widetilde{\mathcal{O}}_s(p;\epsilon) = \int d\vec{p}_1 d\vec{p}_2\;\delta^{(3)}\left(p_1^\mu+p_2^\mu-p^\mu\right)\frac{(-1)^s 2^{\frac{3}{2}s-1}}{s!}$$
$$\times\left(2\left|\vec{p}_1\right|\left|\vec{p}_2\right|-2\vec{p}_1\cdot\vec{p}_2\right)^{\frac{s}{2}}e^{is\Theta(\vec{p}_1,\vec{p}_2)}a(p_1,p_2)$$

: Physical spin-s primary in CFT

Solving Constraints

- Define Kernel $\mathcal{M}_{tot} \equiv \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_4$ $\qquad \tan\theta = \frac{p^2}{p^1}$

$$\mathcal{M}_1 \equiv \exp\left[-\alpha^0\left(\frac{1}{p^0}\left(\bar{\alpha}^1 p^1 + \bar{\alpha}^2 p^2\right)\right)\right] \qquad \mathcal{M}_3 \equiv \exp\left[-\alpha^1\bar{\alpha}^1 \log\left(\frac{q}{p^0}\right)\right]$$

$$\mathcal{M}_2 \equiv \exp\left[-\theta\left(\alpha^1\bar{\alpha}^2 - \alpha^2\bar{\alpha}^1\right)\right] \qquad \mathcal{M}_4 \equiv \exp\left[\frac{\pi}{2} i\alpha^2\bar{\alpha}^2\right]\exp\left[\frac{\pi}{4}\left(\alpha^1\bar{\alpha}^2 - \alpha^2\bar{\alpha}^1\right)\right]$$

  - Designed to generate previous canonical transformation

- Spin-s current : $\widetilde{\mathcal{O}}_s(p;\alpha) = \mathcal{M}_{tot}\mathcal{B}_s(p;\alpha)$

$$\bar{\alpha}_\mu p^\mu \widetilde{\mathcal{O}}_s(p;\alpha) = 0$$
$$\bar{\alpha}^\mu \bar{\alpha}_\mu \widetilde{\mathcal{O}}_s(p;\alpha) = 0$$
$$\alpha^\mu \bar{\alpha}_\mu \widetilde{\mathcal{O}}_s(p;\alpha) = s\widetilde{\mathcal{O}}_s(p;\alpha)$$

$\Rightarrow$

$$(-\bar{\alpha}^0 p^0)\mathcal{B}_s(p;\alpha) = 0$$
$$(2\bar{\alpha}^1\bar{\alpha}^2 - \bar{\alpha}^0 f(p;\bar{\alpha}))\mathcal{B}_s(p;\alpha) = 0$$
$$\alpha^\mu\bar{\alpha}_\mu \mathcal{B}_s(p;\alpha) = s\mathcal{B}_s(p;\alpha)$$

$$\widetilde{\mathcal{O}}_s(p;\alpha) = \mathcal{M}_{tot}\left(\mathcal{B}_s^{(1)}(p)\,(\alpha^1)^s + \mathcal{B}_s^{(2)}(p)\,(\alpha^2)^s\right)$$
$$= \mathcal{B}_s^{(1)}(p)\,(\epsilon_\mu\alpha^\mu)^s + \mathcal{B}_s^{(2)}(p)\,\left(\epsilon^*_\mu\alpha^\mu\right)^s$$

$$\epsilon(p) \equiv \frac{1}{\sqrt{2}\,|\vec{p}|}\left(\frac{\vec{p}^2}{\sqrt{-p^\mu p_\mu}}, \frac{p^0 p^1}{\sqrt{-p^\mu p_\mu}} + ip^2, \frac{p^0 p^2}{\sqrt{-p^\mu p_\mu}} - ip^1\right)$$: polarization vector

Solution : spin-s primary operator

- Solution : $\widetilde{\mathcal{O}}_s(p;\alpha) = \mathcal{B}_s^{(1)}(p)\,(\epsilon_\mu\alpha^\mu)^s + \mathcal{B}_s^{(2)}(p)\,\left(\epsilon_\mu^*\alpha^\mu\right)^s$
  - Properties of polarization vector

$$\epsilon(p)\cdot\epsilon(p) = 0 \qquad \epsilon(p)\cdot\epsilon(p) = 0 \qquad p^\mu\cdot\epsilon_\mu(p) = 0$$

$$\Rightarrow \quad \widetilde{\mathcal{O}}_s(p;\alpha) = \widetilde{\mathcal{O}}_s(p;\epsilon^*)\,(\epsilon_\mu\alpha^\mu)^s + \widetilde{\mathcal{O}}_s(p;\epsilon)\,\left(\epsilon_\mu^*\alpha^\mu\right)^s$$

- Physical projected spin-s primary operator
- Fourier transformation ($p^0 > 0$, $p_a^\mu = (|\vec{p}_a|\,,\vec{p}_a)$)

$$\begin{aligned}\widetilde{\mathcal{O}}_s(p;\alpha) &= 2p^0\int d^3x\; e^{-ip\cdot x}\mathcal{O}^s(x;\alpha)\\ &= \int d\vec{p}_1 d\vec{p}_2\; 2\mathcal{J}(\vec{p}_1,\vec{p}_2)\delta^{(3)}\left(p_1^\mu + p_2^\mu - p^\mu\right)\frac{s!}{\Gamma(s+\frac{1}{2})}\\ &\quad\times(\alpha\cdot p_1 + \alpha\cdot p_2)^s P_s^{-\frac{1}{2},-\frac{1}{2}}\left(\frac{\alpha\cdot p_1 - \alpha\cdot p_2}{\alpha\cdot p_1 + \alpha\cdot p_2}\right) a(\vec{p}_1,\vec{p}_2)\end{aligned}$$

: annihilation operator of bi-local field

(Unconstrained) Primary spin-s Current

- (Unconstrained) Primary spin-s operator

$$\widetilde{\mathcal{O}}_s(p;\epsilon) = \int d\vec{p}_1 d\vec{p}_2\; \delta^{(3)}\left(p_1^\mu + p_2^\mu - p^\mu\right) \frac{(-1)^s 2^{\frac{3}{2}s-1}}{s!}$$

$$\times \left(2\left|\vec{p}_1\right|\left|\vec{p}_2\right| - 2\vec{p}_1\cdot\vec{p}_2\right)^{\frac{s}{2}} e^{is\Theta(\vec{p}_1,\vec{p}_2)} a(p_1,p_2)$$

$$\widetilde{\mathcal{O}}_s(p;\epsilon^*) = \int d\vec{p}_1 d\vec{p}_2\; \delta^{(3)}\left(p_1^\mu + p_2^\mu - p^\mu\right) \frac{(-1)^s 2^{\frac{3}{2}s-1}}{s!}$$

$$\times \left(2\left|\vec{p}_1\right|\left|\vec{p}_2\right| - 2\vec{p}_1\cdot\vec{p}_2\right)^{\frac{s}{2}} e^{-is\Theta(\vec{p}_1,\vec{p}_2)} a(p_1,p_2)$$

$$\Theta\left(\vec{p}_1,\vec{p}_2\right) \equiv 2\arctan\left(\frac{2\vec{p}_1\times\vec{p}_2}{\sqrt{2\sqrt{\vec{p}_1^2\vec{p}_2^2} - 2\vec{p}_1\cdot\vec{p}_2}\left(\sqrt{\vec{p}_1^2} - \sqrt{\vec{p}_2^2}\right)}\right)$$

: Equal to θ map (up to sign)

Other Dimension

- $AdS_4 \times S^2/CFT_3$ covariant gauge

  [JY, A. Jevicki, J.P.Rodrigues, R.d.M.Koch, 2014, 1408.1255]:
  - Representation of SO(2,3) for higher spin in the embedding space (5+5 dim) with constraints & Fronsdal Gauge
  - Possible to find SO(2,3) generators in $AdS_4 \times S^2$
  - Too complicated to find map
- $AdS_3 \times S^1/CFT_2$ covariant gauge :
  - Representation of SO(2,2) for higher spin in the embedding space (4+4 dim) with constraints & Fronsdal Gauge
  - Bi-local map
- AdS_3/CFT_2 time-like :
  - Bi-local map

Thank You