Exact results in the STU model

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with Bernard de Wit and Swapna Mahapatra, to appear

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Introduction

Exact results for F-terms

in $N = 2$ supergravity theories from string theory

Quantum entropy function for BPS black holes in $N = 2$

Microstate counting for BPS black holes in $N = 2$

Wilsonian LEEA for type II superstring theory on $M_4 \times CY_3$:

N=2 supersymmetry in four dimensions.

F-terms:

For every integer $g \geq 0$: $\exists$ F-term at precisely g-loop order in string perturbation theory, schematically

$$\int d^4x\, d^4\theta \sum_{g \geq 0} F^{(g)}(X^I)\,(W^2)^g$$

X^I: conical affine special Kähler manifold $(I = 0, \ldots, h^{1,1}$ (or $h^{2,1}$))

Special Kähler geometry. de Wit, van Proeyen

$F^{(0)}(X)$ determines Kähler metric $N_{IJ} = 2\,\mathrm{Im}\left(\partial^2 F^{(0)}/\partial X^I \partial X^J\right)$.

Introduction

Assemble: $g \geq 1$

$$\Omega(X, A) = \sum_{g=1}^{\infty} F^{(g)}(X)\, A^g \quad , \quad A \in \mathbb{C}$$

Special significance of F-terms:

- determine entropy of half-BPS black holes `OSV, 2004`

  Quantum entropy function: partition function in AdS_2 spacetime
  `Ashoke Sen, 1108.3842`

  Path integral, with $F^{(0)}(X)$ and $\Omega(X, A)$ as input
  `Dabholkar, Gomes, Murthy, 1012.0265`

- $F^{(g)}$ topological string amplitudes.

  Actually, the $F^{(g)}$ are not quite holomorphic ($g \geq 1$):

  holomorphic anomaly equation (recursive relation, $g \geq 2$)

  `Bershadsky, Cecotti, Ooguri, Vafa, 1993`

Two perspectives:

- Wilsonian LEEA
- Holomorphic anomaly of TST

Combine.

Wilsonian LEEA encoded in

$$F(X, A) = F^{(0)}(X) + 2i\,\Omega(X, A)$$

- Can we compute $\Omega(X, A) = \sum_{g=1}^{\infty} F^{(g)}(X)\ A^g$?

  Asymptotic series in $A/(X^0)^2$

- Can we obtain an exact expression for $\Omega(X, A)$?

STU model

Resort to models with exact duality symmetries:

imposes severe restrictions on $\Omega(X, A)$.

STU-model   `Sen and Vafa, hep-th/9508064`

- Exact prepotential $F^{(0)}(X) = -\frac{X^1 X^2 X^3}{X^0}$
- Special coordinates $S = -i\,\frac{X^1}{X^0}\;,\; T = -i\,\frac{X^2}{X^0}\;,\; U = -i\,\frac{X^3}{X^0}$
- $\Omega(X^0, S, T, U, A)$
- S-, T-, U duality symmetry: $\Gamma_0(2) \subset SL(2, \mathbb{Z})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad , \quad c = 0 \mod 2$$

STU-model

- Symplectic vector

$$(X^I, F_I(X,A)) \quad , \quad F_I(X,A) = \frac{\partial F(X,A)}{\partial X^I}$$

- S-duality:

$$\begin{aligned}
X^0 &\rightarrow X^{0\,\prime} = \Delta_S\, X^0 \quad , \quad \Delta_S = ic\,S + d\,, \\
S &\rightarrow S' = \frac{a\,S - ib}{ic\,S + d}\,, \\
T &\rightarrow T' = T + \frac{2}{\Delta_S\,(Y^0)^2}\,\frac{\partial \Delta_S}{\partial S}\,\frac{\partial \Omega}{\partial U}\,, \\
U &\rightarrow U' = U + \frac{2}{\Delta_S\,(Y^0)^2}\,\frac{\partial \Delta_S}{\partial S}\,\frac{\partial \Omega}{\partial T}\,.
\end{aligned}$$

STU-model

$$\left(\frac{\partial\Omega}{\partial T}\right)'_{\rm S} = \frac{\partial\Omega}{\partial T}\,, \qquad \left(\frac{\partial\Omega}{\partial U}\right)'_{\rm S} = \frac{\partial\Omega}{\partial U}\,,$$

$$\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm S} - \Delta_{\rm S}^2\,\frac{\partial\Omega}{\partial S} = \frac{\partial\Delta_{\rm S}}{\partial S}\left[-\Delta_{\rm S}\,X^0\,\frac{\partial\Omega}{\partial X^0} - \frac{2}{(X^0)^2}\,\frac{\partial\Delta_{\rm S}}{\partial S}\,\frac{\partial\Omega}{\partial T}\,\frac{\partial\Omega}{\partial U}\right],$$

$$\left(X^0\,\frac{\partial\Omega}{\partial X^0}\right)'_{\rm S} = X^0\,\frac{\partial\Omega}{\partial X^0} + \frac{4}{\Delta_{\rm S}\,(X^0)^2}\,\frac{\partial\Delta_{\rm S}}{\partial S}\,\frac{\partial\Omega}{\partial T}\,\frac{\partial\Omega}{\partial U}\,.$$

where $\Delta_{\rm S} = icS + d$

Non-linear in Ω. Similarly for T and U duality. Triality.

Ω by iterative procedure.

STU-model

Local supersymmetry requires $\Omega(X, A)$ to be holomorphic and homogeneous of degree 2: $A \in \mathbb{C}$

$$\Omega(X, A) = A \left[\gamma \ln \frac{(X^0)^2}{A} + \omega^{(1)}(S, T, U) + \sum_{n=1}^{\infty} \left(\frac{A}{(X^0)^2} \right)^n \omega^{(n+1)}(S, T, U) \right]$$

Note presence of logarithmic term: beyond usual perturbative expansion in $A/(X^0)^2$.

LEEA input data: $\omega^{(1)}(S, T, U) = \omega(S) + \omega(T) + \omega(U)$ with

- $\omega(S) = \frac{1}{64\pi} \ln \vartheta_2(S)$
- $\omega(S') = \omega(S) - 2\gamma \ln \Delta_{\rm S}(S)$ with $\gamma = -\frac{1}{256\pi}$.
- $X^{0\,\prime} = \Delta_{\rm S}(S)\, X^0$

- $A \in \mathbb{C}$: $\gamma \ln \frac{(X^0)^2}{A} + \omega^{(1)}(S, T, U)$
- $A \in \mathbb{R}$: $\gamma \ln \det N_{IJ} + \left(\omega^{(1)}(S, T, U) + c.c\right)$

STU-model:

$$K - \ln |X^0|^2 = -\ln \left[iX^I \bar{F}_I^{(0)} - iF_I^{(0)} \bar{X}^I \right]$$

Symplectic function.

Log area correction

$A \in \mathbb{C}$: $\quad L_{\text{Wilsonian}} \sim \gamma \ln(X^0)^2 \, W^2 + h.c.$

logarithmic correction to BPS entropy for large black holes from duality:

Area law:

$$\mathcal{S}_0 = \frac{A}{4} = i \left(\bar{X}^I F_I^{(0)} - X^I \bar{F}_I^{(0)} \right) |_{\text{attractor}}$$

Correction:

$$\Delta \mathcal{S} = 2 \ln |X^0|^2 |_{\text{attractor}} = 2 \ln \mathcal{S}_0$$

in agreement with `Sen, arXiv:1108.3842`

STU-model

Higher-order contributions $\omega^{(n)}(S, T, U)$ by iterative procedure:

$$\omega^{(2)}(S, T, U) = \frac{1}{\gamma} \frac{\partial \omega}{\partial S} \frac{\partial \omega}{\partial T} \frac{\partial \omega}{\partial U} .$$

$$\omega^{(3)}(S, T, U) = \frac{1}{4\gamma^3} \left(\frac{\partial \omega}{\partial S} \frac{\partial \omega}{\partial T} \frac{\partial \omega}{\partial U} \right)^2 + \frac{1}{2\gamma^2} \left[\frac{\partial^2 \omega}{\partial S^2} \left(\frac{\partial \omega}{\partial T} \frac{\partial \omega}{\partial U} \right)^2 + \frac{\partial^2 \omega}{\partial T^2} \left(\frac{\partial \omega}{\partial U} \frac{\partial \omega}{\partial S} \right)^2 + \frac{\partial^2 \omega}{\partial U^2} \left(\frac{\partial \omega}{\partial S} \frac{\partial \omega}{\partial T} \right)^2 \right] .$$

etc.

All determined in terms of $\omega^{(1)}(S, T, U) = \omega(S) + \omega(T) + \omega(U) \longrightarrow$

$\Omega(X, A)$ as a series.

Exact expression for Ω? $\longrightarrow$ TST

Relation with topological string theory

Subtle relation betweeen Wilsonian LEEA and the topological string partition function (TST).

LEEA:

$$\Omega(X,A) = A\left[\gamma \ln \frac{(X^0)^2}{A} + \omega^{(1)}(S,T,U) + \sum_{n=1}^{\infty} \left(\frac{A}{(X^0)^2}\right)^n \omega^{(n+1)}(S,T,U)\right]$$

TST: free energies $F^{(g)}(Y,\bar{Y})$, symplectic functions.

Relation: `C, de Wit, Mahapatra, 2014`

- Legendre transformation $\Omega(X,A) \to H(\phi,\chi)$

  H: Hesse potential

- New variables $X^I \to Y^I$

The Hesse potential

LEEA: $F(X,A) = F^{(0)}(X) + 2i\,\Omega(X,A)$

- Hesse potential $H(\phi,\chi)$ `Freed 1997`

$$\phi^I = X^I + \bar{X}^I \qquad , \qquad \chi_I = F_I + \bar{F}_I$$

Obtained by Legendre transform with respect to $X^I - \bar{X}^I$:

$$H(\phi,\chi) = 4\mathrm{Im}\left(F^{(0)}(X) + i\,\Omega(X,A)\right) + i\,\chi_I\,(X - \bar{X})^I$$

Significance: symplectic function

- New variables:

$$\begin{pmatrix}\phi^I \\ \chi_I\end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix}X^I \\ F_I(X,A)\end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix}Y^I \\ F_I^{(0)}(Y)\end{pmatrix}$$

Relation between Y^I and (ϕ,χ) only involves $F^{(0)}$.

Evaluating the Hesse potential

In TST variables Y^I: $H(\phi, \chi) = \sum_{n=1}^{\infty} A^n F^{(n)}(Y, \bar{Y}) + \ldots$

Expansion in terms of symplectic functions $F^{(n)}(Y, \bar{Y})$, the topological free energies.

STU model: LEEA

$$\Omega(X, A) = A \left[\gamma \ln \frac{(X^0)^2}{A} + \omega^{(1)}(S, T, U) + \sum_{n=1}^{\infty} \left(\frac{A}{(X^0)^2} \right)^n \omega^{(n+1)}(S, T, U) \right]$$

TST: $\lambda = A/(Y^0)^2$ topological string coupling

$$H = 4A \left[-\gamma \ln |\lambda^2| + h(\lambda, \omega, N^{IJ}) + \bar{h}(\bar{\lambda}, \bar{\omega}, N^{IJ}) \right]$$

h depends holomorphically on ω and λ, but not on S, T, U:

$$N_{IJ} = 2\,\mathrm{Im} \left(\partial^2 F^{(0)}(Y) / \partial Y^I \partial Y^J \right)$$

Exact results

- $$\frac{\partial h}{\partial \bar{S}} = \frac{2\lambda}{(S+\bar{S})^2} D_T h \, D_U h$$

  Covariant derivative $D_T \omega^{(1)} = \frac{\partial \omega^{(1)}}{\partial T} - \frac{2\gamma}{T+\bar{T}}$

  Equivalent to `BCOV hep-th/9309140`

- h series in λ. Consider $h = h(\lambda, \omega, N^{IJ})$ with $\omega = 0$
  `Alim, Yau, Zhou 2015`

  Can rewrite holomorphic anomaly equation as ODE in $\tilde{\lambda} = \frac{\lambda}{(S+\bar{S})(T+\bar{T})(U+\bar{U})}$:

  $$\frac{dh}{d\tilde{\lambda}} = \frac{2\gamma}{\tilde{\lambda}} - \frac{1}{4\,\tilde{\lambda}^2}\left[1 - \sqrt{1 - 16\,\gamma\,\tilde{\lambda}}\,\right]$$

  Exact expression for h when $\omega = 0$!

Exact results

When $\omega \neq 0$:

$h = f(\tilde{\lambda}) + g(\lambda, \omega)$, $\quad g(\lambda, \omega)$ series expansion in λ, $\quad g(\lambda, \omega = 0) = 0$.

This series contains

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)!} \left(\hat{D}_S^n \, I_2(S) \right) \left(\frac{\lambda}{\gamma} \, D_T \, \omega \, D_U \, \omega \right)^{n+2} ,$$

where $I_2(S) = \frac{\partial^2 \omega}{\partial S^2} + \frac{1}{2\gamma} \left(\frac{\partial \omega}{\partial S} \right)^2$, $\quad \hat{D}_S \, \Sigma(S) = \left(\frac{\partial}{\partial S} + \frac{p}{\gamma} \frac{\partial \, \omega(S)}{\partial S} \right) \Sigma(S)$.

Compare with Cohen-Kuznetsov type series (convergent, modular form of weight k)

$$I(S, \Sigma) = \sum_{m=0}^{\infty} \frac{1}{m! \, (k)_m} \left(D_S^m \, f_k(S) \right) \Sigma^m \quad , \quad \Sigma \in \mathbb{C}$$

Pochhammer symbol $(k)_m = \frac{(k+m-1)!}{(k-1)!} \quad , \quad \Sigma \to \frac{\Sigma}{\Delta_S^2}$

Questions

- Exact expression for $g(\lambda, \omega)$?
- Exact result for Ω on effective action side?

Thanks!